Instantons in the Burgers equation

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The instanton solution for the forced Burgers equation is found. This solution describes the exponential tail of the probability distribution function of velocity differences in the region where shock waves are absent; that is, for large positive velocity differences. The results agree with the one found recently by Polyakov, who used the operator product conjecture. If this conjecture is true, then our WKB asymptotics of the Wyld functional integral should be exact to all orders of perturbation expansion around the instanton solution. We also generalized our solution for the arbitrary dimension of the Burgers (KPZ) equation. As a result we found the asymptotics of the angular dependence of the velocity difference probability distribution function. [S1063-651X(96)05110-0]

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There are two complementary views of the turbulence problem. One regards it as kinetics, in which case the time dependence of the velocity probability distribution function (PDF) must be studied. The Wyld functional integral describes the correlation functions of the velocity field in this picture.

Another view is the Hopf (or Fokker-Planck) approach, where the equal time PDF is studied. For the random force distributed as white noise in time, the closed functional equations (the Fokker-Planck equation) can be derived. In the case of thermal noise the Boltzmann distribution can be derived as an asymptotic solution of this equation. In fact, these two methods are closely related, with the Fokker-Planck equation being simply the Schrödinger equation of the field theory given by the Wyld functional integral.

One of the authors [1] reduced the Hopf equations for the full Navier-Stokes equation to the one dimensional functional equation (loop equation). WKB solutions of this equation were studied, leading to the area law for the velocity circulation PDF.

At the same time, there is significant interest in solving the randomly driven Burgers equation with a large scale driving force (see Ref. [9] and references therein). It is believed that the randomly driven Burgers equation can provide us with the first exactly solvable model of a turbulence-like problem.

In the recent paper by Polyakov [2], which attempted to solve this model exactly, a similar (to that of [1]) method of solving the randomly driven Burgers equation was proposed. It reduced the problem of computations of the equation's correlation functions to the solution of a certain partial differential equation. This equation for the velocity difference PDF can be explicitly solved. However, the derivation of the Polyakov equation was based on the conjecture of the existence of the operator product expansion (OPE).

On the other hand, the Wyld functional integral (see, for example, [7]) provides us with a general field theoretic way computating any correlation functions for any stochastic differential equation. However, in almost all the attempts to use the Wyld functional integral, only its direct perturbative expansion around the zero velocity field was used, as in [3].

Numerous examples worked out in field theory in the last two decades show that in many cases the zero value of fields is not a good starting point for perturbative expansions. Instead, we must expand the functional integrals around the solutions of classical equations of motion having finite actions, the so called instantons. Instantons provide us with field configurations which minimize the action, thus dominating the functional integral in a way similar to the saddle point approximation in ordinary integrals. The name instanton was chosen because such a solution usually exists for a finite time interval to avoid having infinite action (i.e., for an instant). The instanton solution worked out for ordinary quantum mechanics coincides with the WKB approximation of the Schrödinger equation, while in field theory instantons provide us with the only method of performing semiclassical approximations. A review of the instanton methods in quantum mechanics and quantum field theory can be found in [4].

To our knowledge, the first attempt to use the classical equations of motion coming out of the Wyld functional integral was made in [5]. Unfortunately, the classical solutions obtained there were just perturbative expansions around the zero velocity field, while the point of the instanton method is to find classical solutions which account for the nonlinear terms in an exact way. In fact, in all known examples of the applications of instantons, such as quantum mechanical tunneling, Yang-Mills theory, nonlinear sigma model, and in the model considered in this paper, the solution is nonperturbative in coupling constant.

As was conjectured in [5] and later proved in [6], the instantons in the turbulence problem provide us with tails of the probability distributions. The instanton in the turbulence problem is not the same as the solution of the original (Navier-Stokes or Burgers) equation in the usual sense. The force is present, and it acts in a self-consistent way, as required by minimization of the action. This force is no longer random, but is adjusted to provide the large fluctuation of velocity field under consideration.

It was shown in [6] that the probability distribution function for the passive scalar advection in the Gaussian velocity field is asymptotically described by an instanton, with spatially homogeneous strain. Here we find the instanton in the

Burgers equation, in the presence of finite viscosity. The result which we obtain in the turbulent limit (vanishing viscosity) coincides with that of Polyakov, which gives an indirect confirmation to his OPE conjecture. However, the instanton method in its present form confirms only the asymptotics of the probability distribution function of observing a velocity difference between two points when this difference is large and positive. Some further work is required to check if the fluctuations around the instanton solution will keep the probability distribution function from [2] intact.

An analog of the instanton solution found here for the randomly driven Navier-Stokes equation is given in [6]. However, its physical meaning there remains unclear.

We start with the randomly driven Burgers equation

$$u_t+uu_x-\nu u_{xx}=f(x,t), \tag{1}$$

where the force $f(x,t)$ is a Gaussian random field with a pair correlation function

$$\langle f(x,t)f(y,t')\rangle=\delta(t-t')\kappa(x-y). \tag{2}$$

The Wyld functional integral has the following form (see, for example, [7]):

$$\begin{aligned}\int \mathcal{D}f\exp\left(-\tfrac{1}{2}\int dx\,dy\,dt\,f(x,t)D(x-y)f(y,t)\right)\\ &=\int \mathcal{D}f\,\mathcal{D}u\,\delta[u_t+uu_x-\nu u_{xx}-f]\exp\left(-\tfrac{1}{2}\int dx\,dy\,dt\,f(x,t)D(x-y)f(y,t)\right)\\ &=\int \mathcal{D}f\,\mathcal{D}u\mathcal{D}\mu\,\exp\left(\imath\int dx\,dt\mu(u_t+uu_x-\nu u_{xx}-f)-\tfrac{1}{2}\int dx\,dy\,dt\,f(x,t)D(x-y)f(y,t)\right)\\ &=\int \mathcal{D}u\,\mathcal{D}\mu\,\exp\left(\imath\int dx\,dt\,\mu(u_t+uu_x-\nu u_{xx})-\tfrac{1}{2}\int dx\,dy\,dt\,\mu(x,t)\kappa(x-y)\mu(y,t)\right).\end{aligned} \tag{3}$$

Here we started with the obvious functional integral for the force, where D is the function inverse to κ. To change the variable of integration from f to u, we inserted into the functional integral the identity

$$\mathcal{N}=\int \mathcal{D}u\,\delta[u_t+uu_x-\nu u_{xx}-f], \tag{4}$$

where $\mathcal{N}$ is just a number. Dropping that number as an unimportant constant, we removed the δ function at the expense of introducing a "conjugated" variable μ, and evaluated the integral over f to arrive at the final expression in Eq. (3).

It is not so obvious that the integral in Eq. (4) is equal to a pure number because of the determinant $\det(\delta f/\delta u)$ arising in its computation. It is possible, however, to prove (4) using the causality argument (see [7]).

Thus the initial problem of computing the correlation functions of Burgers equation is reduced to the field theory with the action

$$S=-\imath\int \mu(u_t+uu_x-\nu u_{xx})+\tfrac{1}{2}\int dx\,dy\,dt\mu(x,t)\times\kappa(x-y)\mu(y,t). \tag{5}$$

Here we are going to study the correlation function

$$\langle\exp\{\lambda_0(u(\rho_0/2)-u(-\rho_0/2))\}\rangle=\int \mathcal{D}u\,\mathcal{D}\mu\,\exp\{\lambda_0(u(\rho_0/2)-u(-\rho_0/2))-S\}, \tag{6}$$

whose Laplace transform gives us the two point probability distribution (see [2]). We will often refer to the expression we have in the exponential as the action

$$S_{\lambda_0}=S-\lambda_0(u(\rho_0/2)-u(-\rho_0/2)). \tag{7}$$

There are no general methods to compute the functional integrals like Eq. (6) exactly. The most straightforward approach would be to expand the exponential in functional integral in powers of the nonlinear term μuu_x. By doing so we will just reproduce the well known Wyld's diagram technique (see [8]). The attempts to use this technique to describe turbulence always failed, because we were interested in the limit $\nu\to 0$ when a nonlinear term dominates the functional integral. The absence of a large parameter makes the task of computing this functional integral by using perturbation theory hopeless.

Nevertheless, if we are interested in computing the large λ_0 behavior of the correlation function in Eq. (6), we can use λ_0 itself as a large parameter. Then the integral will be dominated by its saddle point, or by the solutions of the equations of motion for the action (7). All we have to do is to find those solutions and compute the value of the action $S_{\rm inst}$ on those solutions. The answer will be given by

$$\langle\exp\{\lambda_0(u(\rho_0/2)-u(-\rho_0/2))\}\rangle=\exp\{-S_{\rm inst}(\lambda_0)+S_{\rm inst}(0)\}. \tag{8}$$

If we want, we can then further expand the integral in powers of $1/\lambda_0$ by using the perturbation theory around those solutions. We will call this method the WKB approximation and the solutions instantons, using the names borrowed from quantum field theory.

To that effect, let us write down the equations of motion corresponding to action (7). They are

$$u_t+uu_x-\nu u_{xx}=-\imath\int dy\,\kappa(x-y)\mu(y), \tag{9}$$

$$\mu_t+u\mu_x+\nu\mu_{xx}=-\imath\lambda_0\left\{\delta\left(x-\frac{\rho_0}{2}\right)-\delta\left(x+\frac{\rho_0}{2}\right)\right\}\delta(t). \tag{10}$$

We note that Eqs. (9) and (10) follow from the Burgers equation, yet they are not exactly the Burgers equation. We suspect that these equations (and their generalization for the Navier-Stokes case) are more fundamental than the Burgers and Navier-Stokes equations themselves, at least as far as the turbulence problem is concerned. A thorough study of their solutions may perhaps become another important problem of mathematical physics.

To solve these equations, let us first notice that the only role the right-hand side of Eq. (10) plays is giving the field μ a finite discontinuity at $t=0$. It is also easy to see that $\mu(t)=0$ for $t>0$. This is because μ feels a negative viscosity, so any solution which is nonzero at $t>0$ will become singular. Thus the field μ can be evaluated at $t=-0$ to be

$$\mu(t=-0)=\imath\lambda_0\left\{\delta\left(x-\frac{\rho_0}{2}\right)-\delta\left(x+\frac{\rho_0}{2}\right)\right\} \tag{11}$$

while it is zero at all later moments of time. It is therefore convenient to speak of the field μ propagating backwards in time starting from its initial value given by Eq. (11). Alternatively, one can argue that the integrals in Eq. (3) are defined only for $t<0$. Those arguments use a striking similarity (actually, an exact correspondence) between Eq. (3) and a Feynman path integral for a quantum mechanical system with the coordinates u and momenta μ to define Eq. (6) as a wave function in the momentum representation. Then conditions (11) become obvious.

If we try to propagate Eq. (11) back in time, we discover that we have to deal with two phenomena governed by the second and third terms of Eq. (10). One of them is just a motion of the initial conditions as dictated by the velocity u. The other is the ''smearing'' of the initial δ function distributions in Eq. (11) due to the viscosity.

However, it can be shown by a direct computation that the smearing does not change the value of the action on the instanton as long as the viscosity is not very large. We will construct a solution at the end of this paper which takes into account the viscosity. For now we will just drop the viscosity term to arrive at a simplified equation

$$\mu_t+u\mu_{xx}=0. \tag{12}$$

Since all this equation can do is moving the δ-function-like singularities around (and changing their heights by compressing them), it is clear that the solution of Eq. (12) with the boundary conditions given by Eq. (11) is just

$$\mu(t)=\imath\lambda(t)\left\{\delta\left(x-\frac{\rho(t)}{2}\right)-\delta\left(x+\frac{\rho(t)}{2}\right)\right\}, \tag{13}$$

with the boundary conditions

$$\lambda(0)=\lambda_0,\quad \rho(0)=\rho_0. \tag{14}$$

Now let us leave Eq. (10) for a while and study Eq. (9). A natural thing to do is to substitute Eq. (13) into the right-hand side of Eq. (9). We obtain

$$u_t+uu_x-\nu u_{xx}=\lambda\left\{\kappa\left(x-\frac{\rho}{2}\right)-\kappa\left(x+\frac{\rho}{2}\right)\right\}. \tag{15}$$

To proceed further, we need to know κ. Let us assume, following [2], that $\kappa(x)$ is a slowly varying even function of x which behaves as

$$\kappa(x)\approx\kappa(0)-\frac{\kappa_0}{2}x^2,\quad |x|\ll\left(\frac{\kappa(0)}{\kappa_0}\right)^{1/2}\equiv L, \tag{16}$$

and quickly turns into zero when $|x|\gg L$. The interval L characterizes the range of the random force and we will work only there; that is, we suppose that ρ also lies within this interval. It is clear then that the contribution to action (7) comes only from the interval L [compare with Eq. (13)]. So we do not have to know the velocity beyond that interval. There we use Eq. (16) to obtain

$$u_t+uu_x-\nu u_{xx}=\lambda\rho\kappa_0 x. \tag{17}$$

Notice that $\kappa(0)$ dropped out.

Equation (17) is a Burgers equation with a linear force. It is easy to solve such an equation. We have to look for the solution in terms of a linear function

$$u(x,t)=\sigma(t)x, \tag{18}$$

which leads to

$$\frac{d\sigma}{dt}+\sigma^2=\kappa_0\lambda\rho. \tag{19}$$

Notice that the viscosity term did not contribute. That does not mean that the viscosity is not important at all. For $x\gg L$ the force in Eq. (15) becomes zero, and the viscosity there could be important. However, in the region $x\propto\rho$, which is the one we study, the viscosity term can be dropped.

Now we can use Eqs. (18) and (13) to solve Eq. (12). A direct substitution leads to

$$\frac{d\lambda}{dt}=\lambda\sigma,$$
$$\frac{d\rho}{dt}=\rho\sigma. \tag{20}$$

These can be solved in terms of the function

$$R(t)=\exp\left(\int_0^t dt'\,\sigma(t')\right)$$

to give

$$\lambda=\lambda_0 R,$$

$$\rho=\rho_0 R, \tag{21}$$

while R itself satisfies, by virtue of Eq. (19), the equation

$$\frac{d^2R}{dt^2}=\kappa_0\rho_0\lambda_0R^3. \tag{22}$$

The last equation has to be solved with the boundary condition $R(-\infty)=0$; otherwise action (7) will not be finite. The solution is given by

$$R=\frac{1}{1-\sqrt{\dfrac{\kappa_0\rho_0\lambda_0}{2}}t}. \tag{23}$$

So we have found the instanton solution for Eqs. (9) and (10). Notice that it is the *only* solution of the equations of motion with given boundary conditions, so we do not have to sum over different instantons. We would like to comment that due to the Galilean invariance of Eqs. (9) and (10), we can always perform a Galilean transformation on the instanton solution to obtain another instanton solution. However, those other solutions will have a nonzero velocity in the infinite past, that is, they describe different ground states of the field theory we consider and we have to discard them (remember, the Galilean symmetry is spontaneously broken; see [2]).

Now it is a matter of a simple computation to find the action on the instanton. We collect everything together and substitute Eqs. (23), (21), (13), and (18) back to Eq. (7) to obtain

$$S_{\rm inst}=-\frac{\sqrt{2\kappa_0}}{3}(\lambda_0\rho_0)^{3/2}, \tag{24}$$

while the correlation function we have been studying is

$$\langle\exp\{\lambda_0(u(\rho_0/2)-u(-\rho_0/2))\}\rangle=\exp\left(\frac{\sqrt{2\kappa_0}}{3}(\lambda_0\rho_0)^{3/2}\right). \tag{25}$$

This is the same answer as the one obtained in [2]. We want to emphasize, however, that we obtained it without any conjectures, and only as an asymptotics for $|\lambda_0|\gg 1$. That allows us to find the asymptotics of the probability distribution of observing the velocity difference u at a distance ρ_0 as

$$P(u,\rho_0)=\left\langle\delta\left\{u-\left[u\left(\frac{\rho_0}{2}\right)-u\left(-\frac{\rho_0}{2}\right)\right]\right\}\right\rangle$$

$$\approx\exp\left\{-\frac{2}{3\kappa_0}\frac{u^3}{\rho_0^3}\right\}, \tag{26}$$

with $u\to+\infty$. Unfortunately the asymptotics (25) does not allow us to find the left tail of the probability distribution as it is related to small λ_0.

We are not going to discuss the physical implications of Eq. (25), referring instead to papers [2] and [9].

Now we return to the question of why we can drop the viscosity in Eq. (10). That is, we just construct the solution of Eq. (10) with the viscosity. To do that, it is convenient to Fourier transform it (taking into account that the velocity u is a linear function of x),

$$\frac{\partial\mu(p)}{\partial t}-\sigma\frac{\partial}{\partial p}(p\mu(p))-\nu p^2\mu=0. \tag{27}$$

Then the solution of Eq. (27) can be found as a direct generalization of Eq. (13):

$$\mu(p)\propto\lambda(t)\sin\left(\frac{p\rho(t)}{2}\right)\exp(-\beta(t)p^2). \tag{28}$$

Here we had to introduce the variable $\beta(t)$ which measures the speed of smearing of the solution with the evident initial condition $\beta(0)=0$. Substituting Eq. (28) into Eq. (27), we reproduce Eq. (21) for ρ and λ with the additional equation for β,

$$\beta_t-2\sigma\beta+\nu=0, \tag{29}$$

with the solution

$$\beta(t)=\frac{\nu}{3\omega}\frac{(1-\omega t)^3-1}{(1-\omega t)^2}, \tag{30}$$

where $\omega=\sqrt{(\kappa_0\rho_0\lambda_0)/2}$.

The smearing, however, has no influence whatsoever on the velocity. To see that, we substitute Eq. (28) into Eq. (9), and arrive back at Eq. (17). In other words, the variable σ still satisfies the same equation (19). A simple argument given below shows that the value of the instanton action depends only on the final value of the velocity of the instanton solution which, as we just showed, does not depend on the viscosity. That is, while Eq. (28) is an exact answer for $\mu(t)$, we can still use Eq. (13) to evaluate the action on the instanton.

We must remember, though, that we should not allow μ to spread beyond the L interval, or more precisely $L>\sqrt{\beta}$. β can become arbitrarily large for large negative times, but the characteristic time interval which contributed to the computation of the instanton action is

$$t_{\rm inst}=\frac{1}{\omega}. \tag{31}$$

So all we have to do is to make sure that $\beta(t_{\rm inst})<L^2$ or

$$L^2>\frac{\nu}{\omega}. \tag{32}$$

This is the condition which viscosity must satisfy for Eq. (25) to be correct and independent of viscosity.

So far we cannot claim that Eq. (25) is an exact answer. It is just a leading asymptotic if $|\lambda_0|$ is a large number. It might be important to estimate the next order contribution to Eq. (25), especially in view of the claim made in [2] that Eq. (25) is actually exact.

To do that it is convenient to introduce the quantity

$$\frac{\partial}{\partial\lambda_0}\ln\langle\exp\{\lambda_0(u(\rho_0/2)-u(-\rho_0/2))\}\rangle = \frac{\int \mathcal{D}u\,\mathcal{D}\mu[u(\rho_0/2)-u(-\rho_0/2)]\exp(-S_{\lambda_0})}{\int \mathcal{D}u\,\mathcal{D}\mu\exp(-S_{\lambda_0})}. \tag{33}$$

It is easy to expand this quantity around the instanton solution. Writing $u=u_{\rm inst}+\tilde u$, $\mu=\mu_{\rm inst}+\tilde\mu$ we arrive for (33) at

$$u_{\rm inst}\left(\frac{\rho_0}{2}\right)-u_{\rm inst}\left(-\frac{\rho_0}{2}\right)+\frac{\int \mathcal{D}\tilde u\mathcal{D}\tilde\mu[\tilde u(\rho_0/2)-\tilde u(-\rho_0/2)]\exp\{-S_{\lambda_0}(u_{\rm inst}+\tilde u,\mu_{\rm inst}+\tilde\mu)\}}{\int \mathcal{D}\tilde u\mathcal{D}\tilde\mu\exp\{-S_{\lambda_0}(u_{\rm inst}+\tilde u,\mu_{\rm inst}+\tilde\mu)\}}. \tag{34}$$

The first term of the expression corresponds to the instanton contribution to the action. We see that it depends only on the value of the instanton solution at $t=0$, justifying our argument that the viscosity does not contribute to the answer. We can easily compute this term by substituting the known instanton solution to obtain

$$\left(\frac{\kappa_0\lambda_0}{2}\right)^{1/2}\rho_0^{3/2}, \tag{35}$$

which is of course compatible with Eq. (25).

To find higher order corrections we have to expand the second term in Eq. (34). It is not difficult to see that if we expand S_{λ_0} $(u_{\rm inst}+\tilde u,\mu_{\rm inst}+\tilde\mu)$ in powers of $\tilde u$ and $\tilde\mu$ up to second order, the contribution from that will be zero, as S will be an even function of $\tilde u$ and $\tilde\mu$. However, due to the presence of the third order terms in the action, it is not clear if the higher order terms are also zero. We intend to investigate this question in another publication.

The analysis of this paper can easily be extended for the case of more than one dimension. The analog of Eqs. (9) and (10) will be

$$\frac{\partial u_i}{\partial t}+\left(u_j\frac{\partial}{\partial x_j}\right)u_i-\nu\Delta u=-\imath\int dy\,\kappa_{ij}(x-y)\mu_j(y), \tag{36}$$

$$\frac{\partial\mu_i}{\partial t}+\frac{\partial}{\partial x_j}(u_j\mu_i)-\mu_j\frac{\partial u_j}{\partial x_i}+\nu\Delta\mu = -\imath\lambda_{0i}\left\{\delta\left(x-\frac{\rho_0}{2}\right)-\delta\left(x+\frac{\rho_0}{2}\right)\right\}\delta(t). \tag{37}$$

The solution of these equations is a direct generalization of Eq. (13), or

$$\mu_i(t)=\imath\lambda_i(t)\left\{\delta\left(x-\frac{\rho(t)}{2}\right)-\delta\left(x+\frac{\rho(t)}{2}\right)\right\}, \tag{38}$$

with

$$\lambda_i(0)=\lambda_{0i},\quad \rho_i(0)=\rho_{0i}. \tag{39}$$

The further progress depends on the tensorial structure of κ_{ij}, which is just a correlation function

$$\langle f_i(x,t)f_j(y,t')\rangle=\kappa_{ij}(x-y)\delta(t-t'). \tag{40}$$

A natural thing to assume would be that the force is a gradient of something; that is, $f_i=\partial_i\Phi$, in which case

$$\kappa_{ij}(x)\approx\kappa_{ij}(0)-\frac{\kappa_0}{6}(x^2\delta_{ij}+2x_ix_j). \tag{41}$$

A direct generalization of the velocity ansatz is

$$u_i=\sigma_{ij}x_j, \tag{42}$$

and Eqs. (19) and (20) turn into

$$\frac{d\lambda_i}{dt}=\lambda_j\sigma_{ji},$$

$$\frac{d\rho_i}{dt}=\sigma_{ij}\rho_j, \tag{43}$$

$$\frac{d\sigma_{ij}}{dt}+\sigma_{ik}\sigma_{kj}=\frac{\kappa_0}{3}(\lambda_i\rho_j+\rho_i\lambda_j+\delta_{ij}\lambda_k\rho_k).$$

σ can actually be eliminated from those equations to give us an analog of Eq. (22),

$$\frac{d^2\lambda_i}{dt^2}=\frac{\kappa_0}{3}(\rho_i\lambda^2+2\lambda_i\lambda_k\rho_k),$$
$$\frac{d^2\rho_i}{dt^2}=\frac{\kappa_0}{3}(\lambda_i\rho^2+2\rho_i\lambda_k\rho_k). \tag{44}$$

While a general solution of those equations is rather difficult to find, it is possible to find the action on the solution by analyzing the corresponding Hamilton-Jacobi equation. To do that, we note that Eqs. (44) are Hamiltonian, with the Hamilton function

$$H=\frac{d\lambda_i}{dt}\frac{d\rho_i}{dt}-\frac{\kappa_0}{6}(\rho^2\lambda^2+2(\lambda_k\rho_k)^2). \tag{45}$$

The (time independent) instanton action S clearly satisfies the equation [10]

$$\frac{\partial S}{\partial\lambda_i}\frac{\partial S}{\partial\rho_i}-\frac{\kappa_0}{6}(\rho^2\lambda^2+2(\lambda_k\rho_k)^2)=0. \tag{46}$$

By rescaling the time and the variables ρ and λ in Eq. (44) we can show that action S has the initial condition dependence

$$S=-\left(\frac{\kappa_0}{6}\right)^{1/2}(\rho_0\lambda_0)^{3/2}f(\cos\varphi), \tag{47}$$

where φ is the angle between the vectors of initial conditions λ_{0i} and ρ_{0i}. This ansatz coincides with the one dimensional answer (24) up to a nontrivial function of the angle $f(\cos\varphi)$, which we would like to determine. Plugging the ansatz into the Hamilton-Jacobi equation, we obtain the equation for $f(\cos\varphi)$,

$$\tfrac{9}{4}zf^2+3ff'(1-z^2)+f'^2(-z+z^3)=1+2z^2, \tag{48}$$

where $z=\cos(\varphi)$. This first order differential equation has to be solved with the boundary condition

$$f(1)=\frac{2}{\sqrt{3}}, \tag{49}$$

which follows directly from Eq. (48) but also can be computed by solving the equation of motion for $\varphi=0$. We can find the function f as a series in powers of $1-z$. It turns out there are two solutions

$$f(z)=\frac{2}{\sqrt{3}}-\frac{\sqrt{3}+\sqrt{11}}{4}(1-z)+\frac{5\sqrt{33}-61}{32(3\sqrt{3}-2\sqrt{11})}(1-z)^2+\cdots, \tag{50}$$

$$f(z)=\frac{2}{\sqrt{3}}+\frac{\sqrt{11}-\sqrt{3}}{4}(1-z)-\frac{5\sqrt{33}+61}{32(3\sqrt{3}+2\sqrt{11})}(1-z)^2+\cdots. \tag{51}$$

Equation (48) does not tell us which of these two to choose. We have to match Eqs. (50) and (51) with the solution of Eq. (44). Those equations cannot be solved in general, but there is a way to find their solution if the angle φ is close to zero, which should be enough to determine $f'(1)$ and therefore to choose the right action.

To do that, we note that the motion represented by Eq. (44) is essentially two dimensional, with all the motion confined to the λ_0,ρ_0 plane. Then we represent λ as two vectors (λ_1,λ_2), while $\rho=(\lambda_1,-\lambda_2)$. Equations (44) turn into (we choose the units where $\kappa_0=3$)

$$\frac{d^2\lambda_1}{dt^2}=3\lambda_1^3-\lambda_1\lambda_2^2,$$
$$\frac{d^2\lambda_2}{dt^2}=-3\lambda_2^3+\lambda_2\lambda_1^2. \tag{52}$$

We choose the boundary conditions $\lambda_0=1$ and $\rho_0=1$. If $\varphi=0$, then $\lambda_2=0$, while λ_1 satisfies the equation

$$\frac{d^2\lambda_1}{dt^2}=3\lambda_1^3; \tag{53}$$

hence

$$\lambda_1=\frac{1}{1-\omega t},\quad \omega=\sqrt{\tfrac{3}{2}}. \tag{54}$$

Now if φ is a small number, then $\lambda_2\ll\lambda_1$, and it satisfies the approximate equation

$$\frac{d^2\lambda_2}{dt^2}=\frac{\lambda_2}{(1-\omega t)^2}, \tag{55}$$

with the solution

$$\lambda_2=\frac{C}{(1-\omega t)^{\alpha}},\quad \alpha=\frac{-3\pm\sqrt{33}}{6}. \tag{56}$$

In particular, for $t=0$,

$$\frac{d\ln\lambda_2}{dt}=\frac{-\sqrt{3}\pm\sqrt{11}}{2\sqrt{2}}. \tag{57}$$

That last quantity can also be evaluated if we know action S. For $t=0$ we obtain

$$\frac{d\rho_i}{dt}=\frac{\partial}{\partial S}\lambda_i=\frac{3\lambda_i}{2\sqrt{2}}f(z)+\frac{f'(z)}{\sqrt{2}}(\rho_i-z\lambda_i), \tag{58}$$

which translates to the language of λ_2 ($\lambda_2\ll\lambda_1$ and $z\approx1$) as

$$\frac{d\ln\lambda_2}{dt}=\sqrt{2}f'(1)-\sqrt{\tfrac{3}{2}}. \tag{59}$$

Comparing with Eq. (57), we obtain

$$f'(1)=\frac{\sqrt{3}\pm\sqrt{11}}{4}. \tag{60}$$

Now we need to choose the plus sign in all the above formulas as we want α to be positive. Otherwise our action will correspond to the solution growing at $t\to-\infty$. That makes us choose the $f(z)$ from Eq. (50), while Eq. (51) has to be discarded.

We would also like to note that, according to Eq. (48), $f(-1)=\imath f(1)$, which can be checked directly by solving the equations of motion at $\varphi=\pi$. Moreover, it can be seen from Eq. (48) that $\imath f(-z)$ is its solution if $f(z)$ is a solution. So we believe there should be some kind of a crossover where the real solution becomes purely imaginary. The fact that the correlation function of a real quantity becomes imaginary should not disturb us. This means that the correlation function we are computing may not exist for a certain value of λ_i, and can only be understood in the sense of analytic continuation. Apparently, the logarithm of the probability distribution function which is obtained by a Legendre transform of the action we found must remain real. One could perform this transform term by term in our expansion.

Summarizing everything, the answer for the D-dimensional case is given by

$$\left\langle \exp\left\{\lambda_i\left[u_i\left(\frac{\rho}{2}\right)-u_i\left(-\frac{\rho}{2}\right)\right]\right\}\right\rangle = \exp\left\{\sqrt{\frac{\kappa_0}{6}}(\rho\lambda)^{3/2}f(\cos\varphi)\right\}, \tag{61}$$

while a Legendre transform of the action will give us the probability distribution function in the form

$$\left\langle \delta\left\{u_i-\left[u_i\left(\frac{\rho}{2}\right)-u_i\left(-\frac{\rho}{2}\right)\right]\right\}\right\rangle \approx \exp\left\{-\frac{2}{3\kappa_0}\frac{u^3}{\rho^3}\left(1+\frac{9(\sqrt{33}-1)}{(9-\sqrt{33})^2}\psi^2+\cdots\right)\right\}, \tag{62}$$

where ψ is the angle between u_i and ρ_i, $\psi\ll 1$, and $u/\rho\to+\infty$.

In conclusion, we would like to say that we have showed by a simple computation that WKB calculations are very useful for understanding the behavior of the randomly driven Burgers equation, and we hope they will be found useful in other problems of turbulence as well. The instanton we found has a spatial homogeneous strain (the velocity was a linear function in the inertial interval) and we suspect it to be a general feature of the instantons in the turbulence problem.

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[10] We would like to note that the Eq. (46) follows from the master equation of [2] if the viscosity is completely neglected. The authors are grateful to A. Polyakov for pointing that out.